STORM HYDROGRAPHS BY TWO-STAGE CONVOLUTION

DETAILED FORM OF COMPUTATIONS

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Convolution is a conventional method of input-output analysis in hydrology (1). Two-stage convolution has been used in hydrologic models in the Southeast Watershed Laboratory (2, 3, 4, 5). The method is useful for both prediction and identification as discussed by Dooge (ibid., p. 11).

Numerous requests have been received for a description of the detailed form of two-stage convolution as used to produce nonlinear response of a watershed to a time-stream of inputs. Several requests were received during the Workshop on Modeling Agricultural Chemicals in the Environment held in Athens, Georgia, in January 1974. Since such computational details would not be acceptable for usual journal publication, this instructional paper was prepared for distribution.

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1. Dooge, J. C. I. Linear Theory of Hydrologic Systems. Tech. Bull. No. 1468. ARS-USDA. Gov. Printing Office, Washington. 1973.

2. Snyder, W. M., W. C. Mills, and J. C. Stephens. A Method of Derivation of Nonconstant Watershed Response Functions. Water Resources Research, Vol. 6, No. 1. 1970.

3. Snyder, W. M., W. C. Mills, and J. C. Stephens. A Three-Component Nonlinear Water-Yield Model. Systems Approach to Hydrology. Proc. First U.S.-Japan Bi-Lateral Seminar in Hydrology. 1971.

4. Snyder, W. M. and L. E. Asmussen. Subsurface Hydrograph Analysis by Convolution. Proc. ASCE, Vol. 98, No. IR3. 1973.

5. Snyder, W. M. Development of a Parametric Hydrologic Model Useful for Sediment Yield. Present and Prospective Technology for Predicting Sediment Yields and Sources. ARS-S-40. 1974.

It is fortuitous that the ARS-sponsored Dooge lectures are now available in published form to provide the theoretical basis for convolutional models. Dooge (ibid., p. 23) gives the convolution of causal systems with continuous data and finite input as in equation 1.

$$y(t) = \int_{0}^{t} x(T)h(t-T)dT \tag{1}$$

y(t) is the continuous time function of output, h(t-T) is the continuous system response (impulse) function, and x(T) is the continuous time function of input. t is real time and T is relative time within the response function.

When input, response, and output are defined at discrete time points, normally at uniform interval, convolution changes from integration to summation as in equation 2.

$$Y(t) = \sum_{T-1}^{t} X(T)\ H\ (t-T+1) \tag{2}$$

Y(t), X(T), and H(t- T+1) are now discrete output, input and response functions. t and T become discrete time variates, which are actually time interval number. H(t- T +1) = 0 for t- T +1 $\leq$ 0 for a causal system.

Equation 2 is a matrix-vector multiplication. Define X(T) as an input vector X_1, X_2, X_3, X_4, and X_5. Define H(t- T +1) as the square matrix in equation 3.

$$H(t-T+1) = \begin{bmatrix} H_1 & 0 & 0 & 0 & 0 \\ H_2 & H_1 & 0 & 0 & 0 \\ H_3 & H_2 & H_1 & 0 & 0 \\ H_4 & H_3 & H_2 & H_1 & 0 \\ H_5 & H_4 & H_3 & H_2 & H_1 \end{bmatrix} \tag{3}$$

With these substitutions, equation 2 becomes equation 4 when expanded.

$$\begin{bmatrix} H_1 & 0 & 0 & 0 & 0 \\ H_2 & H_1 & 0 & 0 & 0 \\ H_3 & H_2 & H_1 & 0 & 0 \\ H_4 & H_3 & H_2 & H_1 & 0 \\ H_5 & H_4 & H_3 & H_2 & H_1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} H_1X_1 \\ H_2X_1 + H_1X_2 \\ H_3X_1 + H_2X_2 + H_1X_3 \\ H_4X_1 + H_3X_2 + H_2X_3 + H_1X_4 \\ H_5X_1 + H_4X_2 + H_3X_3 + H_1X_4 + H_1X_5 \end{bmatrix} = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \end{bmatrix} \quad (4)$$

When using two-stage convolution to construct a storm hydrograph the first stage is used to construct watershed unit response functions. A characteristic function is convolved with a state function. The characteristic function assumes the role of input, the state function is the impulse function and the watershed unit response function is output.

The characteristic function is a time-transformation of a watershed potential runoff map. It is constructed as a histogram of runoff from unit areas vs. time-of-travel classes. A grid system can be superimposed on the watershed to delineate unit areas. Define the characteristic by its histogram values C1, C2, C3, C4, ---, CB. The time-of-travel class width is Δt . Assume that for each interval Δt the watershed acts as a linear reservoir. Then the state function is equation 5.

$$S(T) = A\, e^{-A(T)} \quad (5)$$

A is the time constant of the reservoir.

Snyder and Asmussen (ibid.) show that to change this function for discrete convolution the area under the exponential function is computed for each Δt interval. That is, the integral of equation 5 from $T = t$ to $T = t+1$ is calculated. The successive values for $S(T)$ so computed are given in equations 6.

$$1 - e^{-A} = S(1) \quad (6\text{-}1)$$

$$e^{-A} - e^{-2A} = S(2) \quad (6\text{-}2)$$

$$e^{-2A} - e^{-3A} = S(3) \quad (6\text{-}3)$$

$$e^{-3A} - e^{-4A} = S(4) \quad (6\text{-}4)$$

$$e^{-4A} - e^{-5A} = S(5) \quad (6\text{-}5)$$

Generalize these values to S1(1), S1(2), S1(3), ---, S1(5) to indicate these are the values of the state function for the first interval of time.

Now route each bar or step of the characteristic function, as in equations 7, using the form indicated in equation 4. Only 5 steps are given for illustration, however, the form of calculation is simple and easily extended.

$$S1(1)C1 = q1(1) \quad (7\text{-}1)$$

$$S1(2)C1 + S1(1)C2 = q1(2) \quad (7\text{-}2)$$

$$S1(3)C1 + S1(2)C2 + S1(1)C3 = q1(3) \quad (7\text{-}3)$$

$$S1(4)C1 + S1(3)C2 + S1(2)C3 + S1(1)C4 = q1(4) \quad (7\text{-}4)$$

$$S1(5)C1 + S1(4)C2 + S1(3)C3 + S1(2)C4 + S1(1)C5 = q1(5) \quad (7\text{-}5)$$

In equations 7, we have formed q1(t), the watershed unit response function for the first interval. Any rainfall excess generated in this interval is distributed by q1(t) to form discharge.

The distribution of rainfall excess is second stage convolution. For this second stage, rainfall excess is input, the watershed unit response function is impulse, and stream discharge is output. Perform the first operation (first interval distribution) of the second stage convolution as in equations 8. R1 is rainfall excess for period 1.

$$q1(1)R1 = Q_1 \tag{8-1}$$

$$q1(2)R1 = {}_1Q_2 \tag{8-2}$$

$$q1(3)R1 = {}_1Q_3 \tag{8-3}$$

$$q1(4)R1 = {}_1Q_4 \tag{8-4}$$

$$q1(5)R1 = {}_1Q_5 \tag{8-5}$$

In equations 8, Q_1 is calculated discharge at the end of interval number 1. ${}_1Q_2$ is calculated discharge at end of interval number 2 from rainfall in interval 1. ${}_1Q_3$ is partial discharge at end of interval 2, and so on.

Consider the state or routing function in more detail. To get q1(T) in equations 7, we needed S1(T). These values we got from equations 6, by assuming we had some value of the reservoir time constant, A. Suppose we compute A as in equation 9, making it vary in time but not directly a function of time.

$$A(t) = U + V\ (Q_{t-1} + B_{t-1}) \tag{9}$$

In this equation, Q_{t-1} is discharge at the beginning of each discrete interval, t. B_{t-1} is antecedent base flow, projected under the storm. U and V are regression-type, or empirical, parameters. At the beginning of the first interval storm discharge is zero.

It is well known that watersheds do not act as linear reservoirs. When streams are flowing full, the response to additional rainfall inputs is flashier than when channels are nearly empty. By computing A as in equation 9, using previously calculated discharges as a feedback variable, we can control the flashiness of the watershed, making it act as a nonlinear reservoir. Equation 8-1 gives the discharge at end of interval 1, which

is obviously the discharge at the beginning of interval 2. Specifically, equations 10 give the values of A for the first and second intervals.

$$A(1) = U + V(B_o) \qquad 10\text{-}1$$

$$A(2) = U + V(Q_1 + B_1) \qquad 10\text{-}2$$

In retrospect, A(1) was used in equations 6 to get values of S1(T). This reservoir value feeds through equations 7 and 8 to give stream discharge.

Now moving computationally forward, A(2) will provide values of S2(T) in equations 6. The watershed unit response function for interval 2 is calculated in equations 11.

$$S2(1)C1 = q2(1) \qquad (11\text{-}1)$$

$$S2(2)C1 + S2(1)C2 = q2(2) \qquad (11\text{-}2)$$

$$S2(3)C1 + S2(2)C2 + S2(1)C3 = q2(3) \qquad (11\text{-}3)$$

$$S2(4)C1 + S2(3)C2 + S2(2)C3 + S2(1)C4 = q2(4) \qquad (11\text{-}4)$$

$$S2(5)C1 + S2(4)C2 + S2(3)C3 + S2(2)C4 + S2(1)C5 = q2(5) \qquad (11\text{-}5)$$

Proceeding on, the rainfall excess in the second interval, R2, is distributed by q2(T) in the second stage convolution. Performing this, and combining with equations 8, we get the partial discharges through interval 2, as in equations 12.

$$q1(1)R1 = Q_1 \qquad (12\text{-}1)$$

$$q1(2)R1 + q2(1)R2 = Q_2 \qquad (12\text{-}2)$$

$$q1(3)R1 + q2(2)R2 = {}_2Q_3 \qquad (12\text{-}3)$$

$$q1(4)R1 + q2(3)R2 = {}_2Q_4 \qquad (12\text{-}4)$$

$$q1(5)R1 + q2(4)R2 = {}_2Q_5 \qquad (12\text{-}5)$$

Now we can compute A(3) as in equation 13.

$$A(3) = U + V(Q_2+B_2) \tag{13}$$

Equation 6 gives us S3(T), and we compute q3(T) by the same general form as equations 7 or 11. Again, moving forward, we distribute the rainfall excess of the third interval, R3, and add to what is already computed, as in equations 14.

$$q1(1)R1 = Q_1 \tag{14-1}$$

$$q1(2)R1 + q2(1)R2 = Q_2 \tag{14-2}$$

$$q1(3)R1 + q2(2)R2 + q3(1)R3 = Q_3 \tag{14-3}$$

$$q1(4)R1 + q2(3)R2 + q3(2)R3 = {}_3Q_4 \tag{14-4}$$

$$q1(5)R1 + q2(4)R2 + q3(3)R3 = {}_3Q_5 \tag{14-5}$$

It can be seen that now A(4) can be computed, response functions calculated, and then R4 can be distributed. The process now keeps going until all rainfall excess values are distributed.

The computations outlined above can be performed by numerical methods other than convolution. The reason for using convolution is that it is an efficient and fast process in high-speed calculation. A few nested DO-loops accomplish all the steps. Keeping our models short, fast, and efficient is a fundamental requirement. We must be able to generate not one storm hydrograph, but hundreds or thousands, under many probabilistic sequences of future rainfall for many resource program alternatives. The desire for extreme precision in calculation of one storm must always be tempered by the knowledge that there will be probabilistic variabilities among the many storms of the future.

A model structured on convolution, causes the old but still live arguments of plots vs. watersheds to assume proper perspective. Consider a plot within a watershed. This plot has runoff potential which affects the characteristic function in a watershed. The plot simply "maps into" the characteristic function histogram, along with all other subdrainages of the watershed. The plot will, specifically, affect some ordinates of the characteristic function. First stage convolution, equations 7 and 11 for example, shows how these affected ordinates would affect the response functions. In second stage convolution, the plot effect simply follows through in the formation of the discharge hydrograph.

If a plot were changed so that its runoff potential would change, then a new characteristic function is developed. Beyond this, the plot-change simply follows through the computations of two-stage convolution to give the changed storm hydrograph. While the details of functional formulations are open to question, the basic computational perspective of plots within watersheds is not.

EPA is interested in projecting from watersheds to basins. The same computational system presented above applies just as well in watersheds vs. basins as in plots vs. watersheds. Again details of formulation must be worked out, but the greater computational scheme must hold. For example, one detail would be to incorporate the great amount already known about hydrodynamic channel routing.

APPENDIX

Given in storage:

Characteristic function C(I),I=1,NORD

Rainfall excess vector R(I),I=1,IDUR

Base flow at beginning of storm BO

Base flow under storm B(I),I=1,NORD

Number of ordinates NORD

Number of rainfall intervals IDUR

Length of interval DELT hours

Compute:

State functions SF(I,J),I=1,NORD,J=1,IDUR

Unit response functions UQ(I,J),I=1,NORD,J=1,IDUR

Storm discharge Q(I),I=1,NORD

Total discharge TQ(I),I=1,NORD+1

```
      DO 1000 JB=1,NORD
      DO 1000 JA=1,IDUR
1000  UQ(JB,JA)=0.0
      A=U+V*BO
      ASTART=1.0
      DO 1001 IT=1,NORD
      AEX=-A*IT*DELT
      AJB=EXP(AEX)
      SF(IT,1)=ASTART-AJB
```

```
1001   ASTART=AJB
       DO 1002 KD=1,NORD
       DO 1002 JB=KD,NORD
       JBI=JB-KD+1
1002   UQ(JB,1)=UQ(JB,1)+SF(JBI,1)*C(KD)
       DO 1003 JB=1,NORD
1003   Q(JB)=UQ(JB,1)*R(1)
       DO 1004 ID=2,IDUR
       A=U+V*(Q(ID-1)+B(ID-1))
       ASTART=1.0
       JBEND=NORD-ID+1
       DO 1005 IT=1,JBEND
       AEX=-A*IT*DELT
       AJB=EXP(AEX)
       SF(IT,ID)=ASTART-AJB
1005   ASTART=AJB
       DO 1006 KD=1,JBEND
       DO 1006 JB=KD,JBEND
       JBI=JB-KD+1
1006   UQ(JB,ID)=UQ(JB,ID)+SF(JBI,ID)*C(KD)
       DO 1007 JB=ID,NORD
       JBI=JB-ID+1
1007   Q(JB)=Q(JB)+UQ(JBI,ID)*R(ID)
```

```
1004    CONTINUE
        TQ(1)=BO
        TN=NORD+1
        DO 1008 JB=2,TN
1008    TQ(JB)=Q(JB-1)+B(JB-1)
```